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## Commentary on “Expansions for eigenfunctions and eigenvalues of large- $n$ Toeplitz matrices”

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The paper by L. P. Kadanoff [1] is concerned with the problem of describing the asymptotics of the eigenvalues and eigenvectors of Toeplitz matrices,

$$T_n(\phi) = (\phi_{j-k})_{j,k=0}^{n-1},$$

as the matrix size  $n$  goes to infinity. Here,  $\phi_k$  are the Fourier coefficients of the generating function  $\phi$ , i.e.,

$$\phi(z) = \sum_{k=-\infty}^{\infty} \phi_k z^k, \quad |z| = 1.$$

The concrete focus of the paper are the specific symbols

$$a(z) = (2 - z - 1/z)^\alpha (-z)^\beta, \quad |z| = 1,$$

with  $0 < \alpha < |\beta| < 1$  being real parameters. The function  $a(z)$  is smooth (in fact, analytic) except at the point  $z = 1$ , where it has a “mild” zero. Its image describes a simple closed curve in the complex plane as  $z$  passes along the unit circle.

The problem of the asymptotics of the eigenvalues of Toeplitz matrices has a long history and is a multi-faceted and difficult topic. It is closely connected with the asymptotics of the determinants of Toeplitz matrices and thus with the Szegő Limit

Theorem and its generalizations. The reader is advised to consult, e.g. [3] for many more details and references.

For various classes of symbols  $\phi$ , descriptions have been given for the limiting set (in the Hausdorff metric) of the spectrum of  $T_n(\phi)$  as  $n$  goes to infinity. For instance, a result of Widom [5] states that for certain symbols, the eigenvalues of  $T_n(\phi)$  accumulate asymptotically along the curve described by  $\phi(z)$ ,  $|z| = 1$ . The result applies to the symbols considered here, where it is of importance that  $a(z)$  is non-smooth at precisely one point.

Moreover, under certain conditions on  $\phi$ , one variant of the Szegő Limit Theorem states that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(\lambda_k^{(n)}) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi(e^{ix})) dx,$$

where  $f$  is a smooth test function and  $\lambda_k^{(n)}$  are the eigenvalues of  $T_n(\phi)$ .

One should point out that there are other classes of symbols which show a completely different asymptotics of the Toeplitz eigenvalues. For instance, if  $\phi$  is a rational function, then it is proved that the eigenvalues do (in general) accumulate along arcs which lie inside the curve described by  $\phi$ . This case is best understood because there is an explicit formula for the characteristic polynomial of  $T_n(\phi)$ .

Furthermore, if  $\phi$  is a piecewise continuous function with at least two jump discontinuities, then it is conjectured and numerically substantiated that

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“most”, but not all eigenvalues, accumulate along the image. If there is precisely one jump discontinuity, then one expects that all eigenvalues accumulate along the image.

For continuous real-valued symbols, i.e. for Hermitian Toeplitz matrices, the asymptotics of the eigenvalues is again “canonical”, i.e. the eigenvalues accumulate along the image and the above formula holds for continuous test functions  $f$ .

The two afore-mentioned results give some, but limited information about the eigenvalues of the Toeplitz matrices. The paper under consideration (together with a preceding paper [4]) makes a significant first attempt to determine the asymptotics of the *individual* eigenvalues of  $T_n(a)$ . The asymptotics are obtained up to third order and can be described by

$$\lambda_k^{(n)} = a\left(e^{-ip_k^{(n)}}\right),$$

$$p_k^{(n)} = 2\pi\frac{k}{n} - i(1+2\alpha)\frac{\ln n}{n} + \frac{1}{n}d\left(\frac{k}{n}\right) + o\left(\frac{1}{n}\right),$$

as  $n \rightarrow \infty$ , with an explicit expression for  $d = d(x)$ , which is continuous in  $0 < x < 1$ . (We made some slight changes regarding notation and formulation in comparison with the main formula (25) in [1].) The asymptotics holds uniformly in  $k$  under the assumption  $0 < \varepsilon \leq k/n \leq 1 - \varepsilon$ . The latter means that the description does not catch the eigenvalues accumulating near the point  $0 = a(1)$  where the curve  $a(z)$  is not smooth. At this point, perhaps a different, more complicated asymptotics holds.

The derivation of the results in the paper is not completely rigorous, despite the arguments being quite convincing. The methods are appropriate for dealing with Toeplitz systems. For instance, it is made use of the fact that the finite matrices  $T_n(\phi)$  are naturally related to two semi-infinite Toeplitz systems  $T(\phi) = (\phi_{j-k})$  and  $T(\tilde{\phi}) = (\phi_{k-j})$ ,  $j, k \geq 0$ . In the paper, this is reflected by the use of the auxiliary functions  $\phi^-$  and  $\phi^+$ . The symbol  $\phi$  is equal to  $K(z) = a(z) - \lambda$ , where  $\lambda$  is an eigenvalue (which is to be determined). The two semi-infinite systems are analyzed by Wiener-Hopf factorization, the factors of which serve as approximations for the auxiliary functions  $\phi^-$  and  $\phi^+$ . Both auxiliary functions allow to reconstruct the eigenfunction for  $T_n(\phi)$ .

In view of the argumentation, it seems plausible that the results can be generalized without too much effort to slightly more general symbols,

$$a(z) = (2 - z - 1/z)^\alpha (-z)^\beta b(z), \quad |z| = 1,$$

where  $b(z)$  is a smooth (or analytic) function for  $|z| = 1$  such that  $a(z)$  describes a simple closed curve in the complex plane. On the other hand, notice that symbols with two or more singularities could produce a more complicated eigenvalue behavior [5].

After a preprint version of the paper appeared, Bogoya, Böttcher and Grudsky [2] gave a rigorous proof of the eigenvalue asymptotics in the special case of symbols  $a(z)$  with  $\beta = \alpha - 1$ . The general case is (as of now) still open.

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